Logic

Introduction

Logic is an important concept which has a scope that spans a vast range of disciplines including philosophy, linguistics, computer science, and mathematics. Logic allows us to formulate arguments in a concise manner in order to make deductions and reach conclusions. As such, it is a very useful tool for mathematicians to have. In fact, **George Boole,** an English mathematician who was based in Co. Cork for much of his life, had a keen interest in logic and decided to develop a system for interpreting logical statements in a mathematical way, which eventually became known as Boolean algebra. This branch of algebra involves only two values for variables, true or false, which are subject to three basic operations: AND, OR, and NOT. While there are several additional Boolean operators, they can all be obtained from the composition of these simpler operations. While Boolean algebra was introduced by Boole in the 19th century, the practicality of Boolean algebra did not actually become apparent until a hundred years later when the Information Age began. In fact, it first found a practical application in representing the behaviour of switches in electronic circuits and is now used as the foundation of computer programming. It powers everyday digital devices such as our phones, tablets, laptops, and even search engines like Google! That is part of the beauty of mathematics - while it may sometimes seem an abstract form of ideas and research, the future holds the potential to find important uses for such mathematical discoveries.

Aim of Workshop

The aim of this workshop is to introduce students to the basic Boolean operators and their first practical application in circuit theory. The usefulness of logic as a tool for reasoning will also be emphasised, in addition to its importance in specific areas of mathematics including Boolean algebra.

Learning Outcomes

By the end of this workshop students will be able to:

- Explain the importance of logic in mathematics
- Describe, in their own words, how the AND, OR and NOT operators work
- Describe circuit configurations using Boolean logic statements
- Recognise the technological advances facilitated by Boolean algebra

Keywords

Logic
Logic represents the systematic study of the form of arguments

Logic gate
A logic gate is a circuit that regulates the flow of electrical current in a digital system by taking binary inputs and producing an output under Boolean operations

Logic: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5–10 mins (00:10)	**Activity 1 Who Robbed Kim?**	– Introduce the context of the task. – **Activity Sheet 1:** In pairs, students attempt to solve the mystery of who robbed Kim in Paris (see **Appendix – Note 1**). – Encourage students to draw a table to summarise the suspect's claims.
10 mins (00:20)	**Introduction to Logic Gates**	– Introduce the idea of logic and explain that it is a systematic study of the form of arguments. – Mention that logic has a wide range of applications (See **Workshop Introduction**). – Introduce Boolean algebra and logic gates. Discuss the AND, OR, and NOT gates and draw the symbol for each for students to see (see **Appendix – Note 2**). – You may wish to provide an example of the AND, OR, and NOT gate (see **Appendix – Note 3**).
10 mins (00:30)	**Activity 2 Truth Tables**	– **Activity Sheet 2**: Students fill in activity 2 using the AND, OR and NOT gates (see **Appendix – Note 4**).
10–15 mins (00:45)	**Activity 3 Circuit Tables**	– Describe what is meant by a circuit and explain how it works, making explicit reference to a switch being either open or closed. – **Activity Sheet 3**: Students complete activity 3 using the logic operators (see **Appendix – Note 5**).

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
15 mins (01:00)	**Activity 4 Passcode Riddle**	– **Activity Sheet 4:** In pairs, students try to solve the passcode riddle (see **Appendix – Note 6**). – For the extension question, encourage students to write out all the possible codes that start with 9 AND follow the rules. Now change the first digit to 8 and repeat. Ask students if they notice a pattern after completing several rows. – For Q2, encourage students to write down all possible codes with a product of 36. – If students are stuck, emphasise the fact that the friend didn't know the code after the second clue (suggesting that there are perhaps different codes which have the same sum).
5–10 mins (01:10)	**Boolean Algebra**	– You may wish to mention the use of Boolean algebra in search engines (see **Appendix – Note 7**). – You may also like to discuss Kay McNulty and her contributions to computer programming (see **Appendix – Note 8**).

Logic: Workshop Appendix

Note 1: Solutions to Activity 1

The following table summarises the suspects' claims regarding the robbing of Kim's ring.

	KANYE TELLS US	BEYONCÉ TELLS US	TRUMP TELLS US	MICHAEL D TELLS US	JAY-Z TELLS US
Kanye			✗		✗
Beyoncé	✓			✓	
Trump		✗		✓	
Michael D					✓
Jay-Z	✗	✗	✓		

✓ = robbed Kim   ✗ = didn't robbed Kim

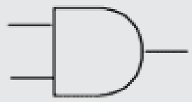
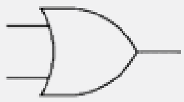
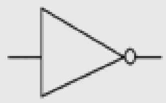
Beyoncé claims that "it wasn't Trump, it wasn't Jay-Z". Since one of these statements is a lie, we know that it must have been either Trump or Jay-Z who committed the crime. However, Michael D tells us that "it was Trump, it was Beyoncé", so we have narrowed it down to three suspects - Trump, Jay-Z or Beyoncé! Trump appears in both of these claims and looking at the rest of the suspects' reports, logic proves that Trump was in fact the one who robbed the ring!

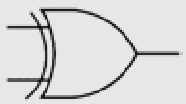
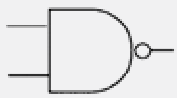
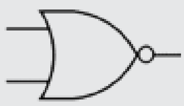
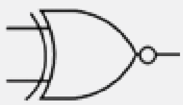
Note 2: Boolean Algebra and Logic Gates

George Boole was a self-taught English mathematician in the 1800s who was interested in extending the applicability of **Aristotle's** philosophical approach to logic. He therefore formulated a system for interpreting logical statements in a mathematical manner, which eventually became known as 'Boolean algebra' in his honour. This form of algebra only concerns the binary variables, true, and false, which are often simply denoted 1 and 0, respectively. These variables are subject to the basic Boolean operators AND, OR and NOT; each of which produces a single output.

Boolean logic is credited with laying the foundations for the information age given that it has been fundamental in the development of digital circuits. Logic gates, in particular, are important components of modern digital systems, which depend on the Boolean operators. These operations allow logic gates to control the flow of current within the system. Logic gates usually have two binary inputs - true and false - and produce a single output. A logic gate is therefore similar to a function in that it takes an input(s) and returns an output.

Different combinations of logic gates enable us to perform more complex operations within the circuit. The 7 basic logic gates are thus AND, OR, NOT, XOR, NAND, NOR, and XNOR, each of which is outlined on the following page. Whilst we will only cover the AND, OR and NOT gate in this workshop, you may still wish to discuss the other logic gates with your students.

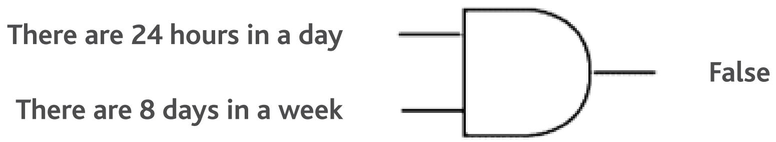
LOGIC GATE	DESCRIPTION	SYMBOL
AND	The AND gate is only true when the two inputs are true. Otherwise, the output is false.	
OR	The OR gate is true if at least one of the inputs is true. Otherwise, the output is false.	
NOT	The NOT gate simply negates the input. For example, if the input is true, the output will be false and vice versa. It is therefore commonly referred to as the inverter gate.	

LOGIC GATE	DESCRIPTION	SYMBOL
XOR	The XOR gate is similar to the OR gate. However, the output is true if either, but not both, of the inputs are true i.e. the output will be false if both inputs are false or if both inputs are true.	
NAND	The NAND gate functions as an AND gate followed by a NOT gate. So, if A was true and B was true, then A AND B would be true. However, since this is followed by a NOT gate, the final output is therefore false.	
NOR	The NOR gate operates as an OR gate followed by a NOT gate.	
XNOR	The XNOR gate is a XOR gate followed by a NOT gate.	

Note 3: Logic Gate Examples

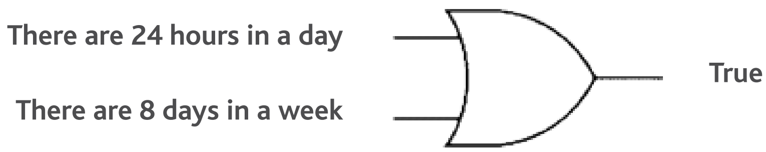
AND Gate

The following AND gate states "there are 24 hours in a day AND there are 8 days in a week". Since both of the claims in this statement are not true, the overall statement is considered false. In other words, since both inputs are not true, the output is therefore false.



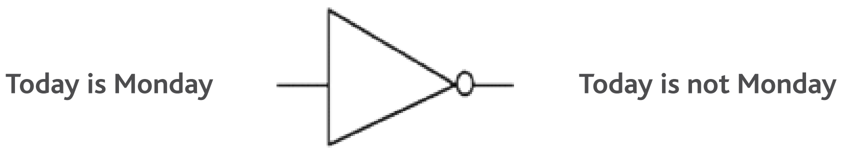
OR Gate

The following OR gate example states "there are 24 hours in a day OR there are 8 days in a week". Since one of these claims is true, the overall statement is considered true. In other words, since at least one of the inputs is true, the output is also true.



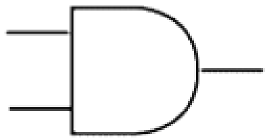
NOT Gate

The NOT gate simply negates the input. Therefore, if the input is "today is Monday", then the output would be "today is not Monday".



Note 4: Solutions for Activity 2

1. (i) What is the symbol for AND?



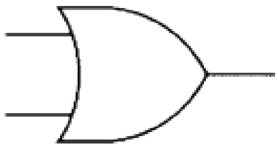
(ii) What is the output for each of the four scenarios below for an AND gate?

	A	B	A AND B
1	TRUE	FALSE	FALSE
2	FALSE	TRUE	FALSE
3	TRUE	TRUE	TRUE
4	FALSE	FALSE	FALSE

(iii) In your own words, how would you define the AND operation?

AND outputs true only when both inputs are true

2. (i) What is the symbol for OR?



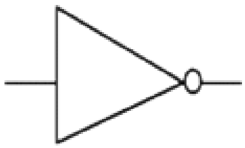
(ii) What is the output for each of the four scenarios below for an OR gate?

	A	B	A OR B
1	TRUE	FALSE	TRUE
2	FALSE	TRUE	TRUE
3	TRUE	TRUE	TRUE
4	FALSE	FALSE	FALSE

(iii) In your own words, how would you define the OR operation?

OR outputs true when either of the two inputs is true

3. (i) What is the symbol for NOT?



(ii) What is the output for each of the two scenarios below for a NOT gate?

	A	NOT A
1	TRUE	FALSE
2	FALSE	TRUE

(iii) In your own words, how would you define the NOT operation?

NOT changes the value of a true input to false and false input to true (i.e. it inverts the input value).

Note 5: Solutions for Activity 3

Q1. (i) Switch A and switch B are said to be in series. Based on your knowledge of a closed circuit, when do you think the bulb will light up?

When A and B are closed

(ii) Summarise this situation using one of the logic operators we mentioned earlier.

A AND B

Q2. Now switch A and switch B are said to be in parallel. Using a logic operator, describe when the bulb will light up in Circuit 2.

A OR B

Q3. (i) Which switch must always be closed for the bulb to light?

Switch R

(ii) In addition to having all of the switches closed, what other combinations of closed switches will turn on the bulb?

a) R and P

b) R and Q

(iii) This situation can be represented by logic gates. Fill in the blanks below with P, Q and R to see how this works.

R AND (P OR Q)

Q4. (i) In addition to having all of the switches closed, what other combinations of closed switches will turn on the bulb?

a) D

b) N_1 and N_2

(ii) Represent this scenario with logic gates (Look back at question 3 (iii) if you're stuck!)

(N_1 AND N_2) OR D

(iii) Fill in the blanks with your answer from the previous question, then complete the table.

N_1	N_2	D	N_1 AND N_2	(N_1 AND N_2) OR D	BULB LIGHT?
T	T	T	T	T	YES
T	T	F	T	T	YES
T	F	T	F	T	YES
F	T	T	F	T	YES
F	F	T	F	T	YES
T	F	F	F	F	NO
F	F	F	F	F	NO

Note 6: Solutions for Activity 4

Q1. How many possible 3-digit codes could be made with 3 digits?

Since there are 10 possible digits (0, 1, ..., 9) for each of the three code numbers we get:

10 x 10 x 10 = 1000

Extension: try this question with all four rules

Method 1:

One way to approach this extension task is to set the first digit of your code equal to 9. Now write down all the possible codes that start with 9 AND obey the four rules. Once you have written down all possible codes, change the first digit to 8 and repeat (see below).

999

888 889 899

777 778 779 788 789 799

666 667 668 669 677 678 679 688 689 699

As we change the first digit of the code for each row, we notice the following pattern emerge:

FIRST DIGIT OF CODE	NO. OF CODES STARTING WITH THE RESPECTIVE DIGIT THAT ALSO OBEY THE FOUR RULES	DIFFERENCE BETWEEN ROWS
9	1	+2
8	3	+3
7	6	+4
6	10	+5
5	15	+6
4	21	+7
3	28	+8
2	36	+9
1	45	+10
0	55	

The values 1, 3, 6, 10, 15, 21, 28, 36, 45, 55 follow the pattern of the triangular sequence. Adding these 10 values together, we get 220 possible codes that obey all four rules.

Method 2:

Alternatively, we could fix the first digit and let the middle digit equal 9. We know there is only one possibility for the third digit in this case, namely 9. Hence, regardless of what the first digit is, there is only one possibility for the third digit when the middle digit is 9.

Continuing this process for a fixed first digit, with 8 as the second digit, there are just two possibilities for the third digit, namely 8 or 9. Thus, we can tabulate all possible 3-digit codes which obey all four rules.

N=	9	8	7	6	5	4	3	2	1	0	SUM
9	1	1	1	1	1	1	1	1	1	1	= 10 X 1 = 10
8		2	2	2	2	2	2	2	2	2	= 9 X 2 = 18
7			3	3	3	3	3	3	3	3	= 8 X 3 = 24
6				4	4	4	4	4	4	4	=7 X 4 = 28
5					5	5	5	5	5	5	= 6 X 5 = 30
4						6	6	6	6	6	= 5 X 6 = 30
3							7	7	7	7	= 4 X 7 = 28
2								8	8	8	= 3 X 8 = 24
1									9	9	= 2 X 9 = 18
0										10	= 1 X 10 = 10
											TOTAL: 220

Also, notice 220 = $\binom{n+2}{3}$ for n = 10, the 10th tetrahedral number.

For more on the tetrahedral numbers see https://en.wikipedia.org/wiki/Tetrahedral_number

Q2. What is the required code?

Of all the positive numbers that have 36 as their product, only two have the same sum (it has to be one of these because otherwise the friend could have cracked it after the second clue!) These are 9 x 2 x 2 and 6 x 6 x 1. Since the largest number only appears once the answer is: 9 x 2 x 2. However, it needs to be written from smallest to largest by code rules: 2 2 9

Note 7: Boolean Algebra and Search Engines

Search engines such as Google or Yahoo employ the use of Boolean logic in order to refine search queries, thereby returning the most relevant pages. The AND operator, for example, will only return results which contain all of the specified terms, whereas the OR operator will return results which contain either one, several or all of the search terms. For instance, if you search "apples or oranges", you will get pages with "apples", pages with "oranges", and also pages with both terms. The NOT operator, on the other hand, will remove any pages with the specified word, allowing you to filter out unwanted results. However, search engines do not actually recognise "not" as the operation, but rather the minus sign character (-). If you are searching for squares but do not want any red squares, for example, then you would have to search "squares -red".

Note 8: Kay McNulty

Whilst Boolean algebra played a significant role in the development of computer programming and digital systems, several Irish mathematicians also made profound contributions to the area. **Kay McNulty**, for example, was an Irish-born mathematician who left Donegal at the age of three to live in Pennsylvania, USA. She graduated with a mathematics degree from Chestnut Hill College in 1942 before securing a job with the U.S. Army research laboratory. This position involved calculating the trajectories for bullets and shells, which was crucial information for soldiers using artillery guns during the war. Later, she was one of six women transferred to work on the E.N.I.A.C. (Electronic Numerical Integrator and Computer) computer; the world's first general purpose digital computer. This mammoth machine was the size of a railway carriage and filled most of the room in which she worked. It had approximately 3,000 switches, 18,000 vacuum tubes, and industrial-sized fans to cool the various components. This huge computer could calculate the trajectories in a matter of seconds, but it was the women's responsibility to use a range of switches and plugs in order to determine the sequence required for these calculations. Unfortunately, they did not have manuals or instructions on how to operate the computer and therefore had to figure out how it worked themselves. Through their work, the women advanced the whole field of computing, with some of the techniques they developed still being used today!

Sources and Additional Resources

http://whatis.techtarget.com/definition/logic-gate-AND-OR-XOR-NOT-NAND-NOR-and-XNOR

https://academo.org/demos/logic-gate-simulator/ (Simulator)

Logic: Activity 1

Logic Times

Bringing you the top activities since 2015 — 3rd April

Who Robbed Kim in Paris

Several suspects have been brought forward regarding the robbing of Kim Kardashian in Paris last week. Each suspect told one truth and one lie, but we don't know which is which. Their claims are reported below:

Image credit: huffingtonpost.com

Kanye: 'It wasn't Jay-Z, it was Beyoncé'

Beyoncé: 'It wasn't Trump, it wasn't Jay-Z'

Trump: 'It was Jay-Z, it wasn't Kanye'

Michael D: 'It was Trump, it was Beyoncé'

Jay-Z: 'It was Michael D, it wasn't Kanye'

Can you figure out who robbed Kim based on the suspects' claims?

Logic: Activity 2

Truth Tables

Q1. AND

(i) What is the symbol for AND?

(ii) What is the output for each of the four scenarios below for an AND gate?

	A	B	A AND B
1	TRUE	FALSE	
2	FALSE	TRUE	
3	TRUE	TRUE	
4	FALSE	FALSE	

(iii) In your own words, how would you define the **AND** operation?

Q2. OR

(i) What is the symbol for OR?

(ii) What is the output for each of the four scenarios below for an OR gate?

	A	B	A OR B
1	TRUE	FALSE	
2	FALSE	TRUE	
3	TRUE	TRUE	
4	FALSE	FALSE	

(iii) In your own words, how would you define the **OR** operation?

Q3. NOT

(i) What is the symbol for NOT?

(ii) What is the output for each of the two scenarios below for a NOT gate?

	A	NOT A
1	TRUE	
2	FALSE	

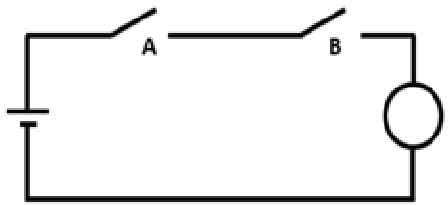
(iii) In your own words, how would you define the **NOT** operation?

The AND, OR and NOT truth tables below will come in handy as you progress through the workshop!

Logic: Activity 3

Using the terms from the previous activity, complete the following questions on circuits.

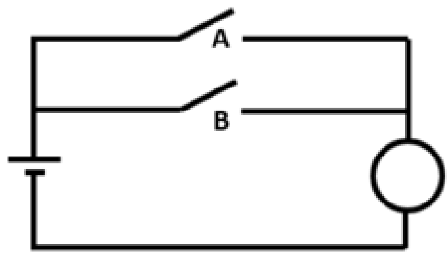
Circuit 1



Q1. (i) Switch A and switch B are said to be *in series*. Based on your knowledge of a closed circuit, when do you think the bulb will light up?

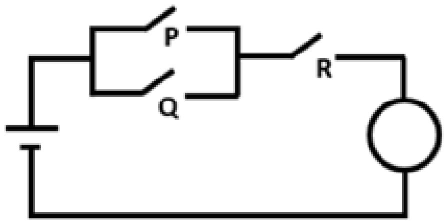
(ii) Summarise this situation using one of the logic gate operators we mentioned earlier.

Circuit 2



Q2. Now switch A and switch B are said to be *in parallel*. Using a logic operator, describe when the bulb will light up in Circuit 2.

Circuit 3



Q3. (i) Which switch must **always** be closed for the bulb to light?

(ii) In addition to having all of the switches closed, what other combinations of closed switches will turn on the bulb?

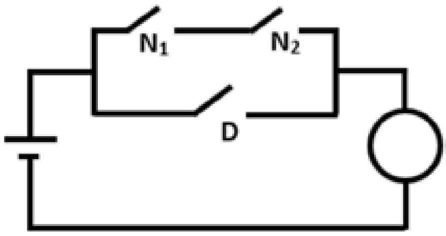
a:________________

b:________________

(iii) This situation can be represented by logic gates. Fill in the blanks with P, Q and R to see how this works.

____ AND (___ OR ___)

Circuit 4



Q4. (i) In addition to having all of the switches closed, what other combinations of closed switches will turn on the bulb?

a:________________

b:________________

(ii) Represent this scenario using logic gates (Look back at Q3 (iii) if you're stuck!)

(iii) The table below lists the possible configurations for the circuit. True corresponds to a closed switch, whilst False means the switch is open. Fill in the blanks with your answer from the previous question, then complete the table.

N_1	N_2	D	N_1 ______ N_2	(N_1 ______ N_2) ______ D	BULB LIGHT?
T	T	T			
T	T	F			
T	F	T			
F	T	T			
F	F	T			
T	F	F			
F	F	F			

(iv) Does your answer in the final column agree with your reasoning in Q4 (i)? Why?

Logic: Activity 4

A safe in a bank is secured with a passcode and protected by a guard. You and your friend need access to the safe. The bored guard decides to help you out with a few logic clues, under the agreement that you each get one guess and that only one of you is led to the safe, whilst the other listens over a walkie talkie. You must both be correct, or the guard will have you removed from the premises.

He firstly outlines the rules of the passcode:

1. A digit is a whole number from 0 to 9
2. The passcode contains 3 digits
3. The second digit is greater than or equal to the first
4. The third digit is greater than or equal to the second

Q1. Taking only the first two rules of the passcode, how many possible codes could be made with 3 digits? (Extension: try this question with all four rules)

You remain at the entrance to the bank with the walkie talkie so that you can hear the conversation of your friend and the guard. Your friend is led through the bank and down one of the corridors to where the safe is kept. She then receives the following clues:

- The product (the three digits multiplied) of the code's digits is 36
- The sum of the code's digits is the same as the corridor number where the safe is held (your friend knows the corridor number, but you do not!)
- The largest of the digits appears only once in the code

Your friend could not crack the code after the second clue. However, when given the third clue, she immediately cracks it and enters in the code. The guard also demands the passcode from you through the walkie talkie.

Q2. What is the required code?

(Hint: Write out all the possible 3-digit codes with a product of 36)